3. Appendix Euler Products

Theorem 3.28 *If f is multiplicative and $D_f(s)$ is absolutely convergent at $s_0 \in \mathbb{C}$ then for all $s : \operatorname{Re} s > \operatorname{Re} s_0$ the Euler Product*

$$\prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + ...\right)$$

converges to $D_f(s)$.

Proof Let $N > 1$ and $\mathcal{N} = \{n : p|n \Rightarrow p \le N\}$. By unique factorisation and because f is multiplicative we have

$$\sum_{n\in\mathcal{N}} \frac{f(n)}{n^s} = \prod_{p\le N} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + ...\right).$$

Thus, for $\operatorname{Re} s > \operatorname{Re} s_0 = \sigma_0$, say,

$$\left|D_f(s) - \prod_{p\le N} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + ...\right)\right|$$

$$= \left|\sum_{n=1}^{\infty} \frac{f(n)}{n^s} - \sum_{n\in\mathcal{N}} \frac{f(n)}{n^s}\right| = \left|\sum_{n\notin\mathcal{N}} \frac{f(n)}{n^s}\right|$$

$$\le \sum_{n\notin\mathcal{N}} \left|\frac{f(n)}{n^s}\right| \le \sum_{n\notin\mathcal{N}} \frac{|f(n)|}{n^{\sigma_0}}.$$

Since we have already seen that $n \le N \Rightarrow n \in \mathcal{N}$ the contrapositive is $n \notin \mathcal{N} \Rightarrow n > N$. Hence

$$\left|D_f(s) - \prod_{p\le N} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + ...\right)\right| \le \sum_{n\notin\mathcal{N}} \frac{|f(n)|}{n^{\sigma_0}}$$

$$\le \sum_{n\ge N+1} \frac{|f(n)|}{n^{\sigma_0}}.$$

We are told that $D_f(s)$ converges absolutely at s_0, therefore

$$\sum_{n=1}^{\infty} \left|\frac{f(n)}{n^{s_0}}\right| = \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma_0}}$$

converges. In particular the tail of this series,

$$\sum_{n \geq N+1} \frac{|f(n)|}{n^{\sigma_0}} \to 0$$

as $N \to \infty$. Hence

$$\prod_{p \leq N} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + \ldots\right)$$

converges to $D_f(s)$ as $N \to \infty$. ∎